MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 4 - *SOLUTIONS*

Problem 1 (20 points)**.** Let (c_n) denote any sequence of real numbers such that $0 < c_n < 1$. Show that if (a_n) is a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = c_n \cdot a_n$, then (a_n) converges. [*Hint*: You won't be able to use the definition of a limit directly, as some of these sequences won't converge to 0! What tools do we have for proving a limit exists without knowing what the limit is?]

Solution. We claim that a_n is decreasing and bounded below. It will then follow that it converges by the monotone convergence theorem. First, we claim that $a_n \geq 0$ for all n. Indeed, this follows by induction, since $a_1 > 0$, and if $a_n > 0$, then $a_{n+1} = c_n \cdot a_n > 0$ since c_n is also positive by assumption. Next, we claim that (a_n) is decreasing. We will show this directly. Indeed, by definition, $a_{n+1} = c_n \cdot a_n < 1 \cdot a_n = a_n$. □

Problem 2 (40 points)**.** Let (a_n) be a sequence of real numbers such that $a_n > 0$ for every $n \in \mathbb{N}$. For each, prove or find a counterexample:

(a) If a_n diverges to ∞, then $1/a_n \to 0$.
(b) If $1/a_n \to 0$, then (a_n) is unbounded.
(c) If (a_n) is bounded above, then $(1/a_n)$ has a convergent subsequence.
(d) If $\liminf a_n > 0$, then $(1/a_n)$ has a convergent subsequence.

Solution. (a) This is true. We wish to show that $1/a_n \to 0$. To this end, fix $\varepsilon > 0$. Since $a_n \to \infty$ and $1/\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, $a_n > 1/\varepsilon > 0$. Therefore, $|a_n| > 1/\varepsilon$ and $\left|\frac{1}{a_n}\right| = \left|\frac{1}{a_n} - 0\right| < \varepsilon$. Hence, $a_n \to 0$.

(b) This is true. Fix $M > 0$. Then $1/M > 0$, so since $1/a_n \to 0$, there exists N such that if $n \geq N$, $|1/a_n| < 1/M$. But then $|a_n| \geq M$, so M cannot bound $|a_n|$.

(c) This is false. Consider the squence defined by $a_n = 1/n$. Then a_n is bounded above, but $1/a_n = n$ diverges to ∞.

(d) This is true. We first claim that if $\liminf a_n > 0$, then a_n is bounded below by a positive real number δ. Indeed, since $\liminf a_n > 0$, we may set $\varepsilon = \liminf a_n$ and choose N such that $\inf\{a_k : k \geq N\} > \dfrac{\liminf a_n}{2}$. Then $\delta = \min\left\{a_1, \ldots, a_{N-1}, \frac{1}{2}\liminf a_n\right\}$ is a lower bound for a_n, and $\delta > 0$ since all terms of a_n are positive by assumption. Then for all terms of (a_n), $0 < 1/a_n \leq 1/\delta$, so (a_n) is a bounded sequence. By the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence.

□

For the next two exercises, we consider the *asymptotic variation* of a sequence (a_n). If (a_n) is a sequence, let $V = \limsup a_n - \liminf a_n$. We assume throughout that $V < \infty$.

Problem 3 (20 points)**.** Show that for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist indices $m, n \in \mathbb{N}$ such that $m, n \geq N$ and $a_m - a_n > V - \varepsilon$.

Solution. Fix $\varepsilon > 0$. Denote $c_n = \sup a_k : k \geq n$ and $b_n = \inf a_k : k \geq n$ be the associated sequences, so that c_n is a decreasing sequence convering to $\limsup a_n$ and b_n is an increasing sequence converging to $\liminf a_n$. Choose N_1 such that if $n \geq N_1$, then $\limsup a_n - c_n < \varepsilon/4$. Choose N_2 such that if $n \geq N_2$, $b_n - \liminf a_n < \varepsilon/4$. Pick $N \geq \max\{N_1, N_2\}$. Then by definition of c_N,, there exists $m \geq N$ such that $a_m > c_N - \varepsilon/4$. Similarly for b_N, there exists $n \geq N$ such that $a_n < b_N + \varepsilon/4$. Then

$$a_m - a_n > c_N - b_N - \varepsilon/2 > (\limsup a_n - \varepsilon/4) - (\liminf a_n + \varepsilon/4) - \varepsilon/2 = V - \varepsilon$$

□

Problem 4 (20 points)**.** Give an example of a sequence such that for any two indices $m, n \in \mathbb{N}$, $a_m - a_n < V$. Justify your answer (ie, calculate V for your sequence and prove that your calculation is correct. Then verify the given property).

Solution. Let $a_n = (-1)^n(1 - 1/n)$. Observe that $|a_n| = |1 - 1/n| \leq 1$. Hence $\limsup a_n \leq 1$ and $\liminf a_n \geq -1$. We claim that these are actually equalities. It suffices to find subsequences of a_n converging to 1 and -1, respectively. Notice that

$$a_{2n} = 1 - 1/(2n) \qquad a_{2n+1} = -(1 - 1/(2n+1)).$$

By the limit arithmetic theorem, $a_{2n} \to 1$ and $a_{2n+1} \to -1$. It follows that $V = \limsup a_n - \liminf a_n = 1 - (-1) = 2$.

Now, we claim that $a_m - a_n < 2$ for all $m, n \in \mathbb{N}$. Indeed, notice that $-1 < a_m, a_n < 1$ for all $m, n \in \mathbb{N}$. Hence $a_m - a_n$ is strictly less than the length of the interval, which is 2. □